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A Trotter–Kato type result for a second order difference inclusion in a Hilbert space

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A B S T R A C T

A Trotter–Kato type result is proved for a class of second order difference inclusions in a real Hilbert space. The equation contains a nonhomogeneous term f and is governed by a nonlinear operator A, which is supposed to be maximal monotone and strongly monotone. The associated boundary conditions are also of monotone type. One shows that, if A^n is a sequence of operators which converges to A in the sense of resolvent and f^n converges to f in a weighted l^2-space, then under additional hypotheses, the sequence of the solutions of the difference inclusion associated to A^n and f^n is uniformly convergent to the solution of the original problem.

1. Introduction

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Consider the boundary value problem

$$\begin{cases} u_{i+1}-(1+\theta_i)u_i+\theta_i u_{i-1}\in c_iAu_i+f_i, & i\geqslant 1,\\ u_1-u_0\in\alpha(u_0-a), & (u_i)_{i\geqslant 1}\in\mathcal{L}, \end{cases} \tag{1.1}$$

where $A:D(A)\subseteq H\to H$ and $\alpha:D(\alpha)\subseteq H\to H$ are nonlinear maximal monotone operators (possible multivalued) in H, with the domains $D(A)$ and $D(\alpha)$, respectively, $a\in H$, $f_i\in H$, $(\forall)i\geqslant 1$ and $c_i>0$, $0<\theta_i<1$, $(\forall)i\geqslant 1$ are given sequences. We have denoted by $\mathcal{L}$ the space $l^2(H)$ with the weight sequence $(\varphi_i)_{i\geqslant 0}$ defined through

$$\varphi_0=1,\qquad \varphi_i=\frac{1}{\theta_1\theta_2\dots\theta_i},\quad i\geqslant 1.$$

Therefore, the scalar product in $\mathcal{L}$ is

$$\langle (u_i)_{i\geqslant 1},(v_i)_{i\geqslant 1}\rangle=\sum_{i=1}^{\infty}\varphi_i(u_i,v_i)$$

and the corresponding norm is

$$\left|(u_i)_{i\geqslant 1}\right|=\left(\sum_{i=1}^{\infty}\varphi_i\|u_i\|^2\right)^{1/2},$$

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whenever the two series converge. Since the sequence $(\varphi_i)_{i\geqslant 0}$ is nondecreasing ($1=\varphi_0\leqslant\varphi_1\leqslant\cdots\leqslant\varphi_i\leqslant\varphi_{i+1}\leqslant\cdots$), the inclusion $\mathcal{L}\subset l^2(H)$ holds both algebraically and topologically.

Existence, uniqueness and asymptotic behavior of the solution for problem (1.1) were investigated in [5,6,22,24], under different hypotheses on A and α. In the case when A and α are subdifferential mappings, problem (1.1) is equivalent with an optimization problem [6].

We work under the following conditions, which assure the existence of the solution of problem (1.1).

(1) $A:D(A)\subseteq H\to H$, $\alpha:D(\alpha)\subseteq H\to H$ are nonlinear maximal monotone operators in H, $0\in D(A)\cap D(\alpha)$, $0\in\alpha(0)$.

(2) A is strongly monotone, i.e. $(\exists)\omega>0$ such that

$$(y_1-y_2,x_1-x_2)\geqslant\omega\|x_1-x_2\|^2,\quad (\forall)x_1,x_2\in D(A),\ (\forall)y_i\in Ax_i,\ i=1,2.$$

(3) If $A_\lambda=(I-(I+\lambda A)^{-1})/\lambda$ is the Yosida approximation of the operator A, then

$$(A_\lambda x-A_\lambda y,z)\geqslant 0,\quad (\forall)z\in\alpha(x-y),\ \text{with } x-y\in D(\alpha).$$

In this case we say that A is α-monotone.

(4) $0<c\leqslant c_i$, $0<\theta_i<1$, $(\forall)i\geqslant 1$ ($c>0$ is a constant), $(f_i)_{i\geqslant 1}\in\mathcal{L}$, $a\in H$.

The following result is proved in [6]:

Theorem 1.1. *If hypotheses* (1)–(4) *hold, then problem* (1.1) *has a unique solution* $u=(u_i)_{i\geqslant 1}\in\mathcal{L}$.

Recall briefly some definitions and results about maximal monotone operators. The multivalued operator $A:D(A)\subseteq H\to H$ is called monotone in the Hilbert space H if for any $x_1,x_2\in D(A)$ and for any $y_i\in Ax_i$, $i=1,2$, we have $(y_1-y_2, x_1-x_2)\geqslant 0$. The monotone operator A is maximal monotone if, regarded as a subset of $H\times H$, is not properly included in any other monotone set of HxH.

The resolvent of the operator A is the single-valued operator $J_\lambda=(I+\lambda A)^{-1}$, $\lambda>0$, with the domain $D(J_\lambda)=H$. The Yosida approximation of A is defined by $A_\lambda=(I-(I+\lambda A)^{-1})/\lambda$, $\lambda>0$, with $D(A_\lambda)=H$. Then A_λ is monotone, J_λ is nonexpansive and $A_\lambda x\in A(J_\lambda x)$, $(\forall)x\in H$. By the definition of A_λ, we have

$$x=J_\lambda x+\lambda A_\lambda x,\quad (\forall)x\in H,\ (\forall)\lambda>0. \tag{1.2}$$

It is known that if A is a maximal monotone operator on H, then there exists a unique semigroup $\{S(t),\ t\geqslant 0\}$ of nonlinear contractions on $\overline{D(A)}$ whose infinitesimal generator is $-A^0$ (A^0 is the minimal section of A). Actually, there exists a one-to-one correspondence between the class of maximal monotone operators on H and the class of semigroups of nonlinear contractions on closed convex subsets of H (see [13, p. 175]). Further information on maximal monotone operators and their applications to the study of different classes of evolution equations can be found in the monographs [13,15,20].

In the present paper we study the continuous dependence on data for problem (1.1). More exactly, consider a sequence of maximal monotone operators $A^n:D(A^n)\subseteq H\to H$, with the domains $D(A^n)$, $n\in\mathbb{Z}$, $n\geqslant 1$, and the sequences $(a^n)_n\subset H$, $f^n=(f_i^n)_{i\geqslant 1}\subset\mathcal{L}$. Similarly to (1.1), consider the sequence of boundary value problems

$$\begin{cases} u_{i+1}^n-(1+\theta_i)u_i^n+\theta_i u_{i-1}^n\in c_iA^nu_i^n+f_i^n, & i\geqslant 1,\\ u_1^n-u_0^n\in\alpha(u_0^n-a^n), & u^n=(u_i^n)_{i\geqslant 1}\in\mathcal{L}. \end{cases} \tag{1.3}$$

About A^n, f^n, one imposes some conditions which are analogous to (1)–(4). They assure the existence and the uniqueness of the solution $u^n=(u_i^n)_{i\geqslant 1}\in\mathcal{L}$ of (1.3). In addition, assume that $a^n\to a$ in H, $f^n\to f$ in $\mathcal{L}$ and A^n converges to A in the sense of resolvent, i.e.

$$\big(I+\lambda A^n\big)^{-1}\xi\to(I+\lambda A)^{-1}\xi,\quad \text{as } n\to\infty,\ (\forall)\lambda>0,\ (\forall)\xi\in H. \tag{1.4}$$

We prove that $u_i^n\to u_i$ (as $n\to\infty$) in H, uniformly with respect to $i\geqslant 1$.

Problem (1.1) is a discrete variant of the evolution equation

$$\begin{cases} p(t)u''(t)+r(t)u'(t)\in Au(t)+f(t), & t\in[0,\infty),\\ u'(0)\in\alpha(u(0)-a), & u\in\mathcal{L}^2_{\tilde r/p}(0,\infty;H). \end{cases} \tag{1.5}$$

Here $\tilde r:[0,\infty)\to\mathbb{R}$ is a weight function given by

$$\tilde r(t)=\exp\left(\int_0^t\frac{r(s)}{p(s)}\,ds\right),\quad t\in[0,\infty),$$

where $p,r:[0,\infty)\to\mathbb{R}$ belong to $W^{1,\infty}(0,\infty)$, $p(t)\geqslant p_0>0$, $r(t)\geqslant r_0>0$ (p_0, r_0 constants). The space $\mathcal{L}^2_{\tilde r/p}(0,\infty;H)$ is the Hilbert space $L^2(0,\infty;H)$ with the weight $\tilde r/p$. This means that the scalar product in $\mathcal{L}^2_{\tilde r/p}(0,\infty;H)$ is

$$\langle u,v\rangle=\int_0^\infty\frac{\tilde r(t)}{p(t)}\big(u(t),v(t)\big)\,dt$$

and the corresponding norm is

$$|u| = \int_0^\infty \frac{\tilde{r}(t)}{p(t)} \|u(t)\|^2 \, dt,$$

whenever the integrals in the right-hand side are convergent.

The existence and the properties of the solution to problem (1.5) were investigated in [4,12,18,21,23]. In the particular case of the boundary value problem $u''(t) \in Au(t)$, $t \in [0, \infty)$, $u(0) = a \in \overline{D(A)}$, the solution defines a semigroup of nonlinear contractions on $\overline{D(A)}$, denoted by $\{S_{1/2}(t),\ t \geqslant 0\}$. Let $-A^0_{1/2}$ be its infinitesimal generator. Then the unique extension of $A^0_{1/2}$ to a maximal monotone operator, denoted by $A_{1/2}$, is called *the square root of the operator* A [12,13].

For a finite time interval $[0, T]$, the following problem was analyzed in [1]:

$$\begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), & t \in [0, T], \\ u'(0) \in \alpha(u(0) - a), \quad -u'(T) \in \beta(u(T) - b). \end{cases} \tag{1.6}$$

Here α, β are maximal monotone operators in H, with the domains $D(\alpha)$, $D(\beta)$. In the particular case when $\alpha = \beta = \partial j$ is the subdifferential of the convex, proper and lower semicontinuous function $j : H \to \overline{\mathbb{R}}$, $j(x) = +\infty$, for $x \neq 0$, $j(0) = 0$, we have $D(\partial j) = \{0\}$ and $\partial j(0) = H$. Therefore, the boundary conditions from (1.6) become $u(0) = a$, $u(T) = b$. For such a problem, the continuous dependence of the solution u on A, a, b, f was proved in [7]. This is a Trotter–Kato type theorem. More exactly, let u^n and u be the solution of (1.6) corresponding to $\{A^n, a^n, b^n\}$ and to $\{A, a, b\}$, respectively. It is shown that if A^n converge to A in the sense of resolvent and $a^n \to a$, $b^n \to b$ in H, then $u^n(t) \to u(t)$ uniformly on compact intervals of t. The goal of this paper is to establish a discrete variant of the above result.

Applications of (1.6) to internal schemes of approximation are presented in [9]. The book [8] contains a detailed study of problems (1.5), (1.6) and their discrete variants, together with some applications to asymptotic approximations. For different types of convergences and hypertopologies of functions, sets and operators, the reader may consult [2,3,10]. Trotter–Kato results for different classes of semigroups of operators are proved in [11,14,16,17,19].

The structure of the paper is the following. In Section 2, we state the hypotheses for our work and establish the main result. Its proof is based on some boundedness lemmas and on some a priori estimates, which are presented in Section 3. To this end, auxiliary boundary value problems are associated and studied. Last section contains an application to partial-difference equations.

2. The hypotheses and the main result

In this section we state the main result of the paper, concerning the convergence of the solution $u^n = (u^n_i)_{i \geqslant 1}$ of problem (1.3) to the solution $u = (u_i)_{i \geqslant 1}$ of problem (1.1) in H, uniformly with respect to $i \in \mathbb{Z}$, $i \geqslant 1$.

We work under the following hypotheses:

$(H1)$ $0 < c \leqslant c_i \leqslant c^*$, $0 < \theta_i < 1$, $(\forall) i \geqslant 1$ ($c, c^* > 0$ are given constants), $f = (f_i)_{i \geqslant 1} \in \mathcal{L}$, $f^n = (f^n_i)_{i \geqslant 1} \in \mathcal{L}$, $a, a^n \in H$.

$(H2)$ $\alpha : D(\alpha) \subseteq H \to H$ are nonlinear maximal monotone operators in H, $0 \in D(\alpha)$, $0 \in \alpha(0)$.

$(H3)$ $A : D(A) \subseteq H \to H$, $A^n : D(A^n) \subseteq H \to H$ are maximal monotone operators in H with $0 \in D(A^n) \cap D(A)$. In addition, suppose that A and A^n are strongly monotone, i.e. $(\exists)\omega > 0$ such that

$$(y_1 - y_2, x_1 - x_2) \geqslant \omega \|x_1 - x_2\|^2,$$

$(\forall) x_1, x_2 \in D(A)$, $(\forall) y_i \in Ax_i$, $i = 1, 2$, and also $(\forall) x_1, x_2 \in D(A^n)$, $(\forall) y_i \in A^n x_i$, $i = 1, 2$.

$(H4)$ A and A^n are α-monotone. This means that, if $A_\lambda = (I - (I + \lambda A)^{-1})/\lambda$ and $A^n_\lambda = (I - (I + \lambda A^n)^{-1})/\lambda$ are the Yosida approximations of the operators A and A^n, respectively, then

$$(A_\lambda x - A_\lambda y, z) \geqslant 0, \qquad \left(A^n_\lambda x - A^n_\lambda y, z\right) \geqslant 0, \quad (\forall) z \in \alpha(x - y), \text{ with } x - y \in D(\alpha).$$

Hypotheses $(H1)$–$(H4)$ assure the existence and uniqueness of the solutions of problems (1.1) and (1.3).

In the sequel we impose some additional conditions concerning the convergence of the data:

$(H5)$ $a^n \to a$ in H, $f^n \to f$ in $\mathcal{L}$, where $f = (f_i)_{i \geqslant 1}$, $f^n = (f^n_i)_{i \geqslant 1}$.

$(H6)$ Sequence A^n converges to A in the sense of resolvent, i.e.

$$\left(I + \lambda A^n\right)^{-1} \xi \to (I + \lambda A)^{-1} \xi, \quad \text{as } n \to \infty, \ (\forall) \lambda > 0, \ (\forall) \xi \in H.$$

The convergence result may now be stated.

Theorem 2.1. *If hypotheses $(H1)$–$(H6)$ hold and $u = (u_i)_{i \geqslant 1}$, $u^n = (u^n_i)_{i \geqslant 1}$ are the solutions of the boundary value problems (1.1) and (1.3) respectively, then $u^n_i \to u_i$ in H as $n \to \infty$, uniformly with respect to i.*

The proof of the theorem combines some ideas from [16] and [7] with some techniques used in the proof of the existence of the solutions [6]. For this purpose, we introduce some auxiliary boundary value problems. Let A^n_λ and A_λ be the Yosida approximations of A^n and A. Consider the problems

$$\begin{cases} w_{i+1}^{\lambda} - (1+\theta_i)w_i^{\lambda} + \theta_i w_{i-1}^{\lambda} \in c_i A w_i^{\lambda} + f_i, \quad i \geqslant 1, \\ w_1^{\lambda} - w_0^{\lambda} \in \alpha(w_0^{\lambda} - y_{\lambda}), \quad w^{\lambda} = (w_i^{\lambda})_{i\geqslant 1} \in \mathcal{L}, \end{cases} \tag{2.1}$$

$$\begin{cases} v_{i+1}^{\lambda} - (1+\theta_i)v_i^{\lambda} + \theta_i v_{i-1}^{\lambda} = c_i A_{\lambda} v_i^{\lambda} + f_i, \quad i \geqslant 1, \\ v_1^{\lambda} - v_0^{\lambda} \in \alpha(v_0^{\lambda} - y_{\lambda}), \quad v^{\lambda} = (v_i^{\lambda})_{i\geqslant 1} \in \mathcal{L}, \end{cases} \tag{2.2}$$

$$\begin{cases} w_{i+1}^{n\lambda} - (1+\theta_i)w_i^{n\lambda} + \theta_i w_{i-1}^{n\lambda} \in c_i A^n w_i^{n\lambda} + f_i^n, \quad i \geqslant 1, \\ w_1^{n\lambda} - w_0^{n\lambda} \in \alpha(w_0^{n\lambda} - y_{n\lambda}), \quad w^{n\lambda} = (w_i^{n\lambda})_{i\geqslant 1} \in \mathcal{L}, \end{cases} \tag{2.3}$$

$$\begin{cases} v_{i+1}^{n\lambda} - (1+\theta_i)v_i^{n\lambda} + \theta_i v_{i-1}^{n\lambda} = c_i A_{\lambda}^n v_i^{n\lambda} + f_i^n, \quad i \geqslant 1, \\ v_1^{n\lambda} - v_0^{n\lambda} \in \alpha(v_0^{n\lambda} - y_{n\lambda}), \quad v^{n\lambda} = (v_i^{n\lambda})_{i\geqslant 1} \in \mathcal{L}, \end{cases} \tag{2.4}$$

where $y_{\lambda} = (I + \sqrt{\lambda}A)^{-1}a$ and $y_{n\lambda} = (I + \sqrt{\lambda}A^n)^{-1}a$. By hypothesis $(H6)$, it follows that

$$y_{n\lambda} \to y_{\lambda}, \qquad A_{\lambda}^n \xi \to A_{\lambda}\xi, \quad \text{as } n \to \infty, \ (\forall)\lambda > 0, \ (\forall)\xi \in H. \tag{2.5}$$

In view of Theorem 1.1, all problems (2.1)–(2.4) admit unique solutions. For every $n, i \in \mathbb{Z}$, $n, i \geqslant 1$, and $\lambda > 0$, we can write

$$\|u_i^n - u_i\| \leqslant \|u_i^n - w_i^{n\lambda}\| + \|w_i^{n\lambda} - v_i^{n\lambda}\| + \|v_i^{n\lambda} - v_i^{\lambda}\| + \|v_i^{\lambda} - w_i^{\lambda}\| + \|w_i^{\lambda} - u_i\|. \tag{2.6}$$

One shows that each term tends to 0 as $n \to \infty$ and $\lambda \to 0$. To do this, we first prove some boundedness results for the solutions of problems (1.3), (2.3), (2.4). Next one obtains some estimates for the terms in the right-hand side of (2.6).

3. The proof of the main result

We start with the boundedness with respect to n of the solution $w^{n\lambda}$ of problem (2.3).

Lemma 3.1. *Under the hypotheses of Theorem* 2.1, *for any fixed $\lambda > 0$, the solution $w^{n\lambda} = (w_i^{n\lambda})_{i\geqslant 1}$ of problem* (2.3) *is bounded with respect to n, uniformly in $i \geqslant 1$. More exactly, the following estimates hold*:

$$\limsup_{n\to\infty} \|w_i^{n\lambda}\| \leqslant C\big(\|y_{\lambda}\| + 1\big), \quad (\forall) i \geqslant 1,$$

$$\limsup_{n\to\infty} \sum_{i=1}^{\infty} \varphi_i \|w_i^{n\lambda}\|^2 \leqslant C\big(\|y_{\lambda}\|^2 + 1\big).$$

Proof. Consider the auxiliary boundary value problem

$$\begin{cases} w_{i+1}^{n\lambda\mu} - (1+\theta_i)w_i^{n\lambda\mu} + \theta_i w_{i-1}^{n\lambda\mu} = c_i A_{\mu}^n w_i^{n\lambda\mu} + \mu w_i^{n\lambda\mu} + f_i^n, \quad i \geqslant 1, \\ w_1^{n\lambda\mu} - w_0^{n\lambda\mu} \in \alpha(w_0^{n\lambda\mu} - y_{n\lambda}), \quad w^{n\lambda\mu} = (w_i^{n\lambda\mu})_{i\geqslant 1} \in \mathcal{L}, \end{cases} \tag{3.1}$$

for arbitrary $\mu > 0$, where $A_{\mu}^n = (I - (I + \mu A^n)^{-1})/\mu$ is the Yosida approximation of A^n. Theorem 1.1 assures the existence of a unique solution $(w_i^{n\lambda\mu})_{i\geqslant 1} \in \mathcal{L}$ for this problem.

We multiply the equation from (3.1) by $\varphi_i w_i^{n\lambda\mu}$ and sum up from $i = 1$ to $i = \infty$. With the aid of the equality $\varphi_i\theta_i = \varphi_{i-1}$, one derives that

$$\begin{aligned} &\sum_{i=1}^{\infty} \varphi_i \big(w_{i+1}^{n\lambda\mu} - w_i^{n\lambda\mu}, w_i^{n\lambda\mu}\big) - \sum_{i=1}^{\infty} \varphi_{i-1} \big(w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu}, w_{i-1}^{n\lambda\mu}\big) - \sum_{i=1}^{\infty} \varphi_{i-1} \|w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu}\|^2 \\ &\quad = \sum_{i=1}^{\infty} c_i\varphi_i \big(A_{\mu}^n w_i^{n\lambda\mu}, w_i^{n\lambda\mu}\big) + \mu \sum_{i=1}^{\infty} \varphi_i \|w_i^{n\lambda\mu}\|^2 + \sum_{i=1}^{\infty} \varphi_i \big(f_i^n, w_i^{n\lambda\mu}\big). \end{aligned} \tag{3.2}$$

Without loss of generality, we may assume that $0 \in A^n 0$. If this condition is not fulfilled, we can replace $A^n w_i^{n\lambda\mu}$ by $A^n w_i^{n\lambda\mu} - (A^n)^0 0$ and f_i^n by $f_i^n + c_i(A^n)^0 0$, where $(A^n)^0 0$ is the element of minimum norm of $A^n 0$. By $(H3)$, we get

$$\begin{aligned} &\sum_{i=1}^{\infty} \varphi_{i-1} \|w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu}\|^2 + \omega \sum_{i=1}^{\infty} c_i\varphi_i \|w_i^{n\lambda\mu}\|^2 + \mu \sum_{i=1}^{\infty} \varphi_i \|w_i^{n\lambda\mu}\|^2 \\ &\quad = -\big(w_1^{n\lambda\mu} - w_0^{n\lambda\mu}, w_0^{n\lambda\mu}\big) - \sum_{i=1}^{\infty} \varphi_i \big(f_i^n, w_i^{n\lambda\mu}\big). \end{aligned}$$

In view of the monotony of α and of the inclusion $0 \in \alpha(0)$, the boundary conditions from (3.1) imply that $(w_1^{n\lambda\mu} - w_0^{n\lambda\mu}, w_0^{n\lambda\mu} - y_{n\lambda}) \geqslant 0$, so

$$-\big(w_1^{n\lambda\mu} - w_0^{n\lambda\mu}, w_0^{n\lambda\mu}\big) \leqslant -\big(w_1^{n\lambda\mu} - w_0^{n\lambda\mu}, y_{n\lambda}\big) \leqslant \left(\sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu} \big\|^2\right)^{1/2} \| y_{n\lambda} \|$$

and thus

$$\sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu} \big\|^2 + \omega c \sum_{i=1}^{\infty} \varphi_i \big\| w_i^{n\lambda\mu} \big\|^2$$
$$\leqslant \left(\sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu} \big\|^2\right)^{1/2} \| y_{n\lambda} \| + \left(\sum_{i=1}^{\infty} \varphi_i \big\| f_i^n \big\|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \varphi_i \big\| w_i^{n\lambda\mu} \big\|^2\right)^{1/2}.$$

Since $f^n = (f_i^n)_{i \geqslant 1}$ is bounded in $\mathcal{L}$, this implies that

$$\sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu} \big\|^2 \leqslant C_1\big(\| y_{n\lambda} \|^2 + 1\big), \tag{3.3}$$

$$\sum_{i=1}^{\infty} \varphi_i \big\| w_i^{n\lambda\mu} \big\|^2 \leqslant C_2\big(\| y_{n\lambda} \|^2 + 1\big), \tag{3.4}$$

where $C_1, C_2 > 0$ are independent of n, λ, μ. Consequently, $w^{n\lambda\mu} = (w_i^{n\lambda\mu})_{i \geqslant 1}$ and $(w_i^{n\lambda\mu} - w_{i-1}^{n\lambda\mu})_{i \geqslant 1}$ are bounded in $\mathcal{L}$ with respect to $\mu > 0$. By (3.1) and (3.4) we also have

$$\sum_{i=1}^{\infty} c_i \varphi_i \big\| A_\mu^n w_i^{n\lambda\mu} \big\|^2 \leqslant C_3\big(\| y_{n\lambda} \|^2 + 1\big), \tag{3.5}$$

$$\big\| w_i^{n\lambda\mu} \big\| \leqslant C_4\big(\| y_{n\lambda} \| + 1\big), \quad i \geqslant 1, \tag{3.6}$$

with $C_3, C_4 > 0$ independent of n, λ, μ, i.

We now show that $w_i^{n\lambda\mu}$ converges in $\mathcal{L}$ to the solution $w_i^{n\lambda}$ of problem (2.3), as $\mu \to 0$. To this end, we write (3.1) for μ and η, multiply their difference by $\varphi_i(w_i^{n\lambda\mu} - w_i^{n\lambda\eta})$ and sum up from $i = 1$ to $i = \infty$:

$$\sum_{i=1}^{\infty} \big[\varphi_i\big(w_{i+1}^{n\lambda\mu} - w_{i+1}^{n\lambda\eta} - w_i^{n\lambda\mu} + w_i^{n\lambda\eta}, w_i^{n\lambda\mu} - w_i^{n\lambda\eta}\big) - \varphi_{i-1}\big(w_i^{n\lambda\mu} - w_i^{n\lambda\eta} - w_{i-1}^{n\lambda\mu} + w_{i-1}^{n\lambda\eta}, w_{i-1}^{n\lambda\mu} - w_{i-1}^{n\lambda\eta}\big)\big]$$
$$= \sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_i^{n\lambda\eta} - w_{i-1}^{n\lambda\mu} + w_{i-1}^{n\lambda\eta} \big\|^2 + \sum_{i=1}^{\infty} c_i \varphi_i \big(A_\mu^n w_i^{n\lambda\mu} - A_\eta^n w_i^{n\lambda\eta}, w_i^{n\lambda\mu} - w_i^{n\lambda\eta}\big)$$
$$+ \sum_{i=1}^{\infty} \varphi_i \big(\mu w_i^{n\lambda\mu} - \eta w_i^{n\lambda\eta}, w_i^{n\lambda\mu} - w_i^{n\lambda\eta}\big).$$

Since $x = J_\mu^n x + \mu A_\mu^n x$ and $x = J_\eta^n x + \eta A_\eta^n x$, $(\forall) x \in H$, it follows that

$$-\big(w_1^{n\lambda\mu} - w_1^{n\lambda\eta} - w_0^{n\lambda\mu} + w_0^{n\lambda\eta}, w_0^{n\lambda\mu} - w_0^{n\lambda\mu}\big)$$
$$= \sum_{i=1}^{\infty} \varphi_{i-1} \big\| w_i^{n\lambda\mu} - w_i^{n\lambda\eta} - w_{i-1}^{n\lambda\mu} + w_{i-1}^{n\lambda\eta} \big\|^2 + \sum_{i=1}^{\infty} c_i \varphi_i \big(A_\mu^n w_i^{n\lambda\mu} - A_\eta^n w_i^{n\lambda\eta}, J_\mu^n w_i^{n\lambda\mu} - J_\eta^n w_i^{n\lambda\eta}\big)$$
$$+ \sum_{i=1}^{\infty} c_i \varphi_i \big(A_\mu^n w_i^{n\lambda\mu} - A_\eta^n w_i^{n\lambda\eta}, \mu A_\mu^n w_i^{n\lambda\mu} - \eta A_\eta^n w_i^{n\lambda\eta}\big) + \sum_{i=1}^{\infty} \varphi_i \big(\mu w_i^{n\lambda\mu} - \eta w_i^{n\lambda\eta}, w_i^{n\lambda\mu} - w_i^{n\lambda\eta}\big). \tag{3.7}$$

Using the boundary conditions of problem (3.1) in μ and η, together with the monotonicity of α, one finds

$$\big(w_1^{n\lambda\mu} - w_1^{n\lambda\eta} - w_0^{n\lambda\mu} + w_0^{n\lambda\eta}, w_0^{n\lambda\mu} - w_0^{n\lambda\mu}\big) \geqslant 0.$$

Since $A_\mu^n x \in A^n(J_\mu^n x)$, $(\forall) x \in H$ and similarly for η, by the strong monotonicity of A^n, we have

$$\big(A_\mu^n w_i^{n\lambda\mu} - A_\eta^n w_i^{n\lambda\eta}, J_\mu^n w_i^{n\lambda\mu} - J_\eta^n w_i^{n\lambda\eta}\big) \geqslant \omega \big\| J_\mu^n w_i^{n\lambda\mu} - J_\eta^n w_i^{n\lambda\eta} \big\|^2,$$

hence

$$\sum_{i=1}^{\infty}\varphi_{i-1}\left\|w_i^{n\lambda\mu}-w_i^{n\lambda\eta}-w_{i-1}^{n\lambda\mu}+w_{i-1}^{n\lambda\eta}\right\|^2+\omega\sum_{i=1}^{\infty}c_i\varphi_i\left\|J_\mu^n w_i^{n\lambda\mu}-J_\eta^n w_i^{n\lambda\eta}\right\|^2$$
$$+\sum_{i=1}^{\infty}c_i\varphi_i\left(\mu\left\|A_\mu^n w_i^{n\lambda\mu}\right\|^2+\eta\left\|A_\eta^n w_i^{n\lambda\eta}\right\|^2\right)+\sum_{i=1}^{\infty}\varphi_i\left(\mu\left\|w_i^{n\lambda\mu}\right\|^2+\eta\left\|w_i^{n\lambda\eta}\right\|^2\right)$$
$$\leqslant(\mu+\eta)\sum_{i=1}^{\infty}c_i\varphi_i\left(A_\mu^n w_i^{n\lambda\mu},A_\eta^n w_i^{n\lambda\eta}\right)+(\mu+\eta)\sum_{i=1}^{\infty}\varphi_i\left(w_i^{n\lambda\mu},w_i^{n\lambda\eta}\right).$$

It follows with the aid of (3.4) and (3.5) that

$$\sum_{i=1}^{\infty}\varphi_{i-1}\left\|w_i^{n\lambda\mu}-w_i^{n\lambda\eta}-w_{i-1}^{n\lambda\mu}+w_{i-1}^{n\lambda\eta}\right\|^2+\omega c\sum_{i=1}^{\infty}\varphi_i\left\|J_\mu^n w_i^{n\lambda\mu}-J_\eta^n w_i^{n\lambda\eta}\right\|^2\leqslant C_5(\mu+\eta)\left(\|y_{n\lambda}\|^2+1\right), \tag{3.8}$$

with $C_5>0$ independent of n, λ, μ, η.

Therefore $w_i^{n\lambda\mu}-w_{i-1}^{n\lambda\mu}$ and $J_\mu^n w_i^{n\lambda\mu}$ are strongly convergent in $\mathcal{L}$ as $\mu\to 0$. Since $w_i^{n\lambda\mu}=J_\mu^n w_i^{n\lambda\mu}+\mu A_\mu^n w_i^{n\lambda\mu}$, with $A_\mu^n w_i^{n\lambda\mu}$ bounded in $\mathcal{L}$ (see (3.5)), we get the convergence of $w_i^{n\lambda\mu}$ in $\mathcal{L}$, say $w_i^{n\lambda\mu}\to w_i^{n\lambda}$ in $\mathcal{L}$, as $\mu\to 0$. In view of the inequality

$$\left\|w_i^{n\lambda\mu}-w_i^{n\lambda}\right\|^2\leqslant\sum_{i=1}^{\infty}\varphi_i\left\|w_i^{n\lambda\mu}-w_i^{n\lambda}\right\|^2,\quad(\forall)i\geqslant 1,$$

we deduce that $w_i^{n\lambda\mu}\to w_i^{n\lambda}$ in H as $\mu\to 0$, uniformly with respect to $i\geqslant 1$. Obviously, $J_\mu^n w_i^{n\lambda\mu}\to w_i^{n\lambda}$ in $\mathcal{L}$ and H.

Inequality (3.5) implies also the boundedness of $A_\mu^n w_i^{n\lambda\mu}$ in H, uniformly with respect to $i\in\mathbb{Z}$, $i\geqslant 1$. Hence $A_\mu^n w_i^{n\lambda\mu}\rightharpoonup g_i^{n\lambda}$ (weakly in H) as $\mu\to 0$. Passing to the limit as $\mu\to 0$ in the inclusion $A_\mu^n w_i^{n\lambda\mu}\in A^n(J_\mu^n w_i^{n\lambda\mu})$, one arrives at $g_i^{n\lambda}\in A^n w_i^{n\lambda}$ and $w_i^{n\lambda}\in D(A^n)$, $(\forall)i\geqslant 1$.

Now we may pass to the limit as $\mu\to 0$ in (3.1) and deduce that $w^{n\lambda}=(w_i^{n\lambda})_{i\geqslant 1}$ is the unique solution of problem (2.3). The boundary conditions from (2.3) follow with the aid of the monotonicity of α. Taking the limit as $\mu\to 0$ in (3.4), (3.6) we get

$$\sum_{i=1}^{\infty}\varphi_i\left\|w_i^{n\lambda}\right\|^2\leqslant C_2\left(\|y_{n\lambda}\|^2+1\right), \tag{3.9}$$

$$\left\|w_i^{n\lambda}\right\|\leqslant C_4\left(\|y_{n\lambda}\|+1\right),\quad(\forall)i\geqslant 1. \tag{3.10}$$

The claim follows via (2.5), by passing to the superior limit as $n\to\infty$ in (3.9) and (3.10).

Analogously we can find estimates for the solutions of the problems (1.3), (2.1), (2.2) and (2.4). □

Lemma 3.2. *Under assumptions* $(H1)$–$(H5)$, *the solution* $u^n=(u_i^n)_{i\geqslant 1}$ *of problem* (1.3) *is bounded in* H, *uniformly with respect to* i. *More exactly,*

$$\limsup_{n\to\infty}\left\|u_i^n\right\|\leqslant C\left(\|a\|+1\right),\quad(\forall)i\geqslant 1. \tag{3.11}$$

Lemma 3.3. *If* $(H1)$–$(H6)$ *are satisfied, then for each* $\lambda>0$, *the solution* $v^{n\lambda}=(v_i^{n\lambda})_{i\geqslant 1}$ *of* (2.4) *is bounded with respect to* n, *uniformly in* $i\geqslant 1$, *namely*

$$\limsup_{n\to\infty}\left\|v_i^{n\lambda}\right\|\leqslant C\left(\|y_\lambda\|+1\right),\quad(\forall)i\geqslant 1, \tag{3.12}$$

$$\limsup_{n\to\infty}\sum_{i=1}^{\infty}\varphi_i\left\|v_i^{n\lambda}\right\|^2\leqslant C\left(\|y_\lambda\|^2+1\right). \tag{3.13}$$

In addition,

$$\limsup_{n\to\infty}\left\|A_\lambda^n v_i^{n\lambda}\right\|\leqslant C\left(\|y_\lambda\|+1\right),\quad(\forall)i\geqslant 1. \tag{3.14}$$

Lemma 3.4. *Under hypotheses* $(H1)$–$(H4)$, *the solution of problem* (2.1) *satisfies the estimate*

$$\left\|w_i^\lambda\right\|\leqslant C\left(\|y_\lambda\|+1\right),\quad(\forall)i\geqslant 1,\ (\forall)\lambda>0. \tag{3.15}$$

In what follows, we analyze the convergence of each term from the right-hand side of (2.6) and (2.7).

Lemma 3.5. *If* $(H1)$–$(H6)$ *hold, then for all* $\lambda > 0$,

$$\limsup_{n\to\infty} \left\| u_i^n - w_i^{n\lambda} \right\| \leqslant C_1 \|a - y_\lambda\|, \quad (\forall) i \geqslant 1. \tag{3.16}$$

Proof. Multiplying the difference between Eqs. (1.3) and (2.3) by $\varphi_i(u_i^n - w_i^{n\lambda})$, we get

$$\sum_{i=1}^{\infty} \varphi_i \big(u_{i+1}^n - w_{i+1}^{n\lambda} - u_i^n + w_i^{n\lambda}, u_i^n - w_i^{n\lambda}\big) - \sum_{i=1}^{\infty} \varphi_{i-1} \big(u_i^n - w_i^{n\lambda} - u_{i-1}^n + w_{i-1}^{n\lambda}, u_i^n - w_i^{n\lambda}\big)$$
$$= \sum_{i=1}^{\infty} c_i \varphi_i \big(A^n u_i^n - A^n w_i^{n\lambda}, u_i^n - w_i^{n\lambda}\big).$$

Here $A^n u_i^n$ and $A^n w_i^{n\lambda}$ denote two arbitrary elements of the sets $A^n u_i^n$ and $A^n w_i^{n\lambda}$, respectively. By the strong monotonicity of A^n, this implies that

$$\sum_{i=1}^{\infty} \varphi_{i-1} \left\| u_i^n - w_i^{n\lambda} - u_{i-1}^n + w_{i-1}^{n\lambda} \right\|^2 + \omega c \sum_{i=1}^{\infty} \varphi_i \left\| u_i^n - w_i^{n\lambda} \right\|^2$$
$$\leqslant -\big(u_1^n - w_1^{n\lambda} - u_0^n + w_0^{n\lambda}, u_0^n - w_0^{n\lambda}\big). \tag{3.17}$$

The boundary conditions from (1.3) and (2.3), together with the monotonicity of α, provide us with the inequality

$$\big(u_1^n - w_1^{n\lambda} - u_0^n + w_0^{n\lambda}, u_0^n - a_n - w_0^{n\lambda} + y_{n\lambda}\big) \geqslant 0,$$

so

$$-\big(u_1^n - w_1^{n\lambda} - u_0^n + w_0^{n\lambda}, u_0^n - w_0^{n\lambda}\big) \leqslant -\big(u_1^n - w_1^{n\lambda} - u_0^n + w_0^{n\lambda}, a_n - y_{n\lambda}\big)$$
$$\leqslant \left(\sum_{i=1}^{\infty} \varphi_{i-1} \left\| u_i^n - w_i^{n\lambda} - u_{i-1}^n + w_{i-1}^{n\lambda} \right\|^2 \right)^{1/2} \|a_n - y_{n\lambda}\|.$$

This, together with (3.17), leads to

$$\sum_{i=1}^{\infty} \varphi_{i-1} \left\| u_i^n - w_i^{n\lambda} - u_{i-1}^n + w_{i-1}^{n\lambda} \right\|^2 \leqslant \|a_n - y_{n\lambda}\|^2$$

and therefore,

$$\left\| u_i^n - w_i^{n\lambda} \right\|^2 \leqslant \sum_{i=1}^{\infty} \varphi_i \left\| u_i^n - w_i^{n\lambda} \right\|^2 \leqslant C \|a_n - y_{n\lambda}\|^2.$$

Passing to the superior limit as $n \to \infty$ and using $(H5)$ and (2.5), one arrives at (3.16). □

Lemma 3.6. *For every* $\lambda > 0$ *and* $i \geqslant 1$, *the solutions of problems* (2.3) *and* (2.4) *verify the estimate*

$$\limsup_{n\to\infty} \left\| w_i^{n\lambda} - v_i^{n\lambda} \right\| \leqslant C_2 \sqrt{\lambda}. \tag{3.18}$$

Proof. As above, we multiply the difference between Eqs. (2.3) and (2.4) by $\varphi_i(w_i^{n\lambda} - v_i^{n\lambda})$ and use the monotonicity of α, to find that

$$\sum_{i=1}^{\infty} \varphi_{i-1} \left\| w_i^{n\lambda} - v_i^{n\lambda} - w_{i-1}^{n\lambda} + v_{i-1}^{n\lambda} \right\|^2 + \sum_{i=1}^{\infty} c_i \varphi_i \big(s_i^{n\lambda} - A_\lambda^n v_i^{n\lambda}, w_i^{n\lambda} - v_i^{n\lambda}\big) \leqslant 0, \tag{3.19}$$

where $s_i^{n\lambda}$ is the element $[w_{i+1}^{n\lambda} - (1+\theta_i) w_i^{n\lambda} + \theta_i w_{i-1}^{n\lambda} - f_i^n]/c_i$ of the set $A^n w_i^{n\lambda}$. Since $s_i^{n\lambda} \in A^n w_i^{n\lambda}$, $A_\lambda^n v_i^{n\lambda} \in A^n(J_\lambda^n v_i^{n\lambda})$ and A^n is a monotone operator, we have

$$\big(s_i^{n\lambda} - A_\lambda^n v_i^{n\lambda}, w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda}\big) \geqslant \omega \left\| w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda} \right\|^2.$$

Introducing this, together with the equality $v_i^{n\lambda} = J_\lambda^n v_i^{n\lambda} + \lambda A_\lambda^n v_i^{n\lambda}$ in (3.19), we get

$$\sum_{i=1}^{\infty} \varphi_{i-1} \| w_i^{n\lambda} - v_i^{n\lambda} - w_{i-1}^{n\lambda} + v_{i-1}^{n\lambda} \|^2 + \omega \sum_{i=1}^{\infty} c_i \varphi_i \| w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda} \|^2 \leqslant \lambda \sum_{i=1}^{\infty} c_i \varphi_i \big(s_i^{n\lambda} - A_\lambda^n v_i^{n\lambda}, A_\lambda^n v_i^{n\lambda} \big).$$

This implies

$$\sum_{i=1}^{\infty} \varphi_{i-1} \| w_i^{n\lambda} - v_i^{n\lambda} - w_{i-1}^{n\lambda} + v_{i-1}^{n\lambda} \|^2 + \omega c \sum_{i=1}^{\infty} \varphi_i \| w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda} \|^2 \leqslant \lambda \sum_{i=1}^{\infty} c_i \varphi_i \| s_\lambda^n \|^2, \quad \lambda > 0,\ n \in \mathbb{N}.$$

But s_λ^n is bounded with respect to n in $\mathcal{L}$ (via Lemma 3.1). So there exists a constant $m > 0$ such that

$$\sum_{i=1}^{\infty} \varphi_i \| w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda} \|^2 \leqslant m\lambda, \quad \lambda > 0,\ n \in \mathbb{N}. \tag{3.20}$$

From the estimate $\| w_i^{n\lambda} - v_i^{n\lambda} \| \leqslant (\sum_{i=1}^{\infty} c_i \varphi_i \| w_i^{n\lambda} - J_\lambda^n v_i^{n\lambda} \|^2)^{1/2} + \lambda \| A_\lambda^n v_i^{n\lambda} \|$, $i \geqslant 1$, $\lambda > 0$, $n \in \mathbb{N}$, it follows in view of (3.20) and (3.14) that

$$\| w_i^{n\lambda} - v_i^{n\lambda} \| \leqslant K \sqrt{\lambda} \big(\| y_\lambda \| + 1 \big),$$

where K is independent of i, n, λ. Since $y_\lambda = (I + \sqrt{\lambda} A)^{-1} a$ is convergent as $\lambda \to 0$, there is a constant $C_2 > 0$ such that (3.18) holds. $\square$

Lemma 3.7. *Under hypotheses* $(H1)$–$(H6)$, *the solutions of problems* (2.2) *and* (2.4) *satisfy the equality*

$$\lim_{n \to \infty} \| v_i^{n\lambda} - v_i^\lambda \| = 0, \quad (\forall)\lambda > 0,\ (\forall) i \geqslant 1. \tag{3.21}$$

Proof. By (2.2) and (2.4), we can easily obtain

$$\begin{aligned} &\sum_{i=1}^{\infty} \varphi_i \big(v_{i+1}^{n\lambda} - v_{i+1}^\lambda - v_i^{n\lambda} + v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big) - \sum_{i=1}^{\infty} \varphi_{i-1} \big(v_i^{n\lambda} - v_i^\lambda - v_{i-1}^{n\lambda} + v_{i-1}^\lambda, v_i^{n\lambda} - v_i^\lambda \big) \\ &\quad = \sum_{i=1}^{\infty} c_i \varphi_i \big(A_\lambda^n v_i^{n\lambda} - A_\lambda v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big) + \sum_{i=1}^{\infty} \varphi_i \big(f_i^n - f_i, v_i^{n\lambda} - v_i^\lambda \big). \end{aligned}$$

It is known from [1] that if A^n is ω-monotone, then A_λ^n is $1/(1+\lambda\omega)$-monotone. This leads to

$$\begin{aligned} \big(A_\lambda^n v_i^{n\lambda} - A_\lambda v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big) &= \big(A_\lambda^n v_i^{n\lambda} - A_\lambda^n v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big) + \big(A_\lambda^n v_i^\lambda - A_\lambda v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big) \\ &\geqslant \frac{1}{1+\lambda\omega} \| v_i^{n\lambda} - v_i^\lambda \|^2 + \big(A_\lambda^n v_i^\lambda - A_\lambda v_i^\lambda, v_i^{n\lambda} - v_i^\lambda \big). \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{i=1}^{\infty} \varphi_{i-1} \| v_i^{n\lambda} - v_i^\lambda - v_{i-1}^{n\lambda} + v_{i-1}^\lambda \|^2 + K \sum_{i=1}^{\infty} \varphi_i \| v_i^{n\lambda} - v_i^\lambda \|^2 \\ &\quad \leqslant -\big(v_1^{n\lambda} - v_1^\lambda - v_0^{n\lambda} + v_0^\lambda, v_0^{n\lambda} - v_0^\lambda \big) \\ &\qquad + \left[\left(\sum_{i=1}^{\infty} c_i^2 \varphi_i \| A_\lambda^n v_i^\lambda - A_\lambda v_i^\lambda \|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} \varphi_i \| f_i^n - f_i \|^2 \right)^{1/2} \right] \left(\sum_{i=1}^{\infty} \varphi_i \| v_i^{n\lambda} - v_i^\lambda \|^2 \right)^{1/2}. \end{aligned} \tag{3.22}$$

The boundary conditions and the monotonicity of α imply

$$\big(v_1^{n\lambda} - v_1^\lambda - v_0^{n\lambda} + v_0^\lambda, v_0^{n\lambda} - v_0^\lambda - y_{n\lambda} + y_\lambda \big) \geqslant 0,$$

so

$$\begin{aligned} -\big(v_1^{n\lambda} - v_1^\lambda - v_0^{n\lambda} + v_0^\lambda, v_0^{n\lambda} - v_0^\lambda \big) &\leqslant \| v_1^{n\lambda} - v_1^\lambda - v_0^{n\lambda} + v_0^\lambda \| \cdot \| y_{n\lambda} - y_\lambda \| \\ &\leqslant \left(\sum_{i=1}^{\infty} \varphi_{i-1} \| v_i^{n\lambda} - v_i^\lambda - v_{i-1}^{n\lambda} + v_{i-1}^\lambda \|^2 \right)^{1/2} \| y_{n\lambda} - y_\lambda \|. \end{aligned}$$

Thus (3.22) leads to

$$\left(\sum_{i=1}^{\infty}\varphi_{i-1}\left\|v_i^{n\lambda}-v_i^{\lambda}-v_{i-1}^{n\lambda}+v_{i-1}^{\lambda}\right\|^2\right)^{1/2}$$
$$\leqslant C_1\left[\|y_{n\lambda}-y_\lambda\|+\left(\sum_{i=1}^{\infty}c_i^2\varphi_i\left\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\right\|^2\right)^{1/2}+\left(\sum_{i=1}^{\infty}\varphi_i\left\|f_i^n-f_i\right\|^2\right)^{1/2}\right]$$

and

$$\left(\sum_{i=1}^{\infty}\varphi_i\left\|v_i^{n\lambda}-v_i^{\lambda}\right\|^2\right)^{1/2}\leqslant C_2\left[\|y_{n\lambda}-y_\lambda\|+\left(\sum_{i=1}^{\infty}c_i^2\varphi_i\left\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\right\|^2\right)^{1/2}+\left(\sum_{i=1}^{\infty}\varphi_i\left\|f_i^n-f_i\right\|^2\right)^{1/2}\right].$$

We can prove now that the series $\sum_{i=1}^{\infty}c_i^2\varphi_i\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\|^2$ is uniformly convergent with respect to n. Indeed, since $\lambda A_\lambda x=x-J_\lambda x$ and $J_\lambda^n 0=J_\lambda 0=0$, where $J_\lambda x=(I+\lambda A)^{-1}x$, we have the estimate

$$\left\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\right\|=\left\|J_\lambda^n v_i^\lambda-J_\lambda v_i^\lambda\right\|/\lambda\leqslant\left[\left\|J_\lambda^n v_i^\lambda-J_\lambda^n 0\right\|+\left\|J_\lambda v_i^\lambda-J_\lambda 0\right\|\right]/\lambda.$$

But J_λ^n and J_λ are nonexpansive mappings. This implies that $\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\|\leqslant 2\|v_i^\lambda\|/\lambda$, for any n and i. Since c_i is bounded, it follows that $\sum_{i=1}^{\infty}c_i^2\varphi_i\|v_i^\lambda\|^2<\infty$, so the series $\sum_{i=1}^{\infty}c_i^2\varphi_i\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\|^2$ is uniformly convergent with respect to n.

Next, we use the second part of (2.5) to conclude that $\sum_{i=1}^{\infty}c_i^2\varphi_i\|A_\lambda^n v_i^\lambda-A_\lambda v_i^\lambda\|^2\to 0$ as $n\to\infty$. By virtue of $(H5)$, one deduces that $\lim_{n\to\infty}\sum_{i=1}^{\infty}\varphi_i\|v_i^{n\lambda}-v_i^\lambda\|^2=0$, $(\forall)\lambda>0$ and since $\|v_i^{n\lambda}-v_i^\lambda\|\leqslant(\sum_{i=1}^{\infty}\varphi_i\|v_i^{n\lambda}-v_i^\lambda\|^2)^{1/2}$, $(\forall)i\geqslant 1$, we arrive at (3.21). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Passing to the superior limit (as $n\to\infty$) in (2.6), we obtain (for each fixed $\lambda>0$) with the aid of Lemmas 3.5–3.7:

$$\limsup_{n\to\infty}\left\|u_i^n-u_i\right\|\leqslant C_1\|a-y_\lambda\|+C_2\sqrt{\lambda}+\left\|v_i^\lambda-w_i^\lambda\right\|+\left\|w_i^\lambda-u_i\right\|,\quad(\forall)\lambda>0,\ (\forall)i\geqslant 1.\tag{3.23}$$

We can also find estimates for the last two terms, which are similar to those from Lemmas 3.5 and 3.6, namely

$$\left\|w_i^\lambda-u_i\right\|\leqslant C_3\|a-y_\lambda\|,\qquad\left\|v_i^\lambda-w_i^\lambda\right\|\leqslant C_4\sqrt{\lambda},\quad(\forall)\lambda>0,\ (\forall)i\geqslant 1,$$

where C_3, $C_4>0$ are independent of λ and i. Therefore, by (3.23) we get

$$\limsup_{n\to\infty}\left\|u_i^n-u_i\right\|\leqslant C_5\|a-y_\lambda\|+C_6\sqrt{\lambda},\quad(\forall)\lambda>0,$$

uniformly with respect to $i\geqslant 1$. Passing to the limit as $\lambda\to 0$ in the above inequality and using the convergence $y_\lambda\to a$ as $\lambda\to 0$, we derive that $u_i^n\to u_i$ in H (as $n\to\infty$), uniformly with respect to $i\geqslant 1$. This completes the proof. □

4. An example

Let Ω be an open bounded subset of $\mathbb{R}^d$, $d\geqslant 1$, with the boundary $\partial\Omega$ smooth enough. We work in the Hilbert spaces $H=L^2(\Omega)$ and $\mathcal{L}=\mathcal{L}(L^2(\Omega))$. Let $\beta:D(\beta)\subseteq\mathbb{R}\to\mathbb{R}$, $\beta^n:D(\beta^n)\subseteq\mathbb{R}\to\mathbb{R}$ be a sequence of maximal monotone operators in $\mathbb{R}$ such that $0\in D(\beta)\cap D(\beta^n)$. Then the operators

$$Au=-\Delta u+\beta(u),$$
$$D(A)=\left\{u\in H^2(\Omega)\cap H_0^1(\Omega),\ (\exists)v\in L^2(\Omega),\ v(x)\in\beta\big(u(x)\big)\text{ a.e. }x\in\Omega\right\}$$

and

$$A^n u=-\Delta u+\beta^n(u),$$
$$D\big(A^n\big)=\left\{u\in H^2(\Omega)\cap H_0^1(\Omega),\ (\exists)v\in L^2(\Omega),\ v(x)\in\beta^n\big(u(x)\big)\text{ a.e. }x\in\Omega\right\}$$

are maximal monotone and strongly monotone in $H=L^2(\Omega)$ [13, p. 89]. If $\beta^n\to\beta$, then A^n converges to A in the sense of the resolvent.

Assume that the operator $\alpha:D(\alpha)\subseteq L^2(\Omega)\to L^2(\Omega)$ is maximal monotone in $L^2(\Omega)$, such that $0\in D(\alpha)$, $0\in\alpha(0)$ and condition (3) holds for both A and A^n. If $0<c\leqslant c_i\leqslant c^*$, $0<\theta_i<1$, $(\forall)i\geqslant 1$ are given sequences and $a,a^n\in L^2(\Omega)$, then we compare the solutions of the boundary value problems

$$\begin{cases}u_{i+1}(x)-(1+\theta_i)u_i(x)+\theta_i u_{i-1}(x)\in-c_i\Delta u_i(x)+\beta(u_i(x)), & x\in\Omega,\\ u_i(x)=0, & x\in\partial\Omega,\\ u_1(x)-u_0(x)\in\alpha(u_0(x)-a(x)), & x\in\Omega,\ (u_i)_{i\geqslant 1}\in\mathcal{L}(L^2(\Omega))\end{cases}\tag{4.1}$$

and

$$\begin{cases} u_{i+1}^n(x) - (1+\theta_i)u_i^n(x) + \theta_i u_{i-1}^n(x) \in -c_i \Delta u_i^n(x) + \beta^n(u_i^n(x)), & x \in \Omega, \\ u_i^n(x) = 0, & x \in \partial\Omega, \\ u_1^n(x) - u_0^n(x) \in \alpha(u_0^n(x) - a^n(x)), & x \in \Omega,\ (u_i^n)_{i \geqslant 1} \in \mathcal{L}(L^2(\Omega)). \end{cases} \tag{4.2}$$

As a consequence of Theorem 1.1, we first remark that these problems admit unique solutions $u = (u_i)_{i \geqslant 1}$, $u^n = (u_i^n)_{i \geqslant 1} \in \mathcal{L}$. By virtue of Theorem 2.1, we have the following continuous dependence on data result.

Corollary 4.1. *Suppose that in addition to the above hypotheses, $a^n \to a$ in $H = L^2(\Omega)$ and $\beta^n \to \beta$ in the sense of the resolvent. Then, $u_i^n \to u_i$ as $n \to \infty$ in H, uniformly with respect to i.*

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